On Partitioning the n-Cube into Sets with Mutual Distance 1

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Abstract

We consider a problem mentioned in [1], which is in partitioning the n-cube in as many sets as possible, such that two different sets always have distance one.

1 Introduction and approach

Assume that the vertices of the n-cube are partitioned in such a way that the Hamming distance between each pair of the subsets in the partition is 1. What is the maximal number of such subsets? Denoting this number by $m(n)$, one can easily check the following table:

n:	0	1	2	3	4
$m(n)$:	1	2	3	4	8

Only the case $n = 4$ may raise some difficulties. In this case the 4-cube must be partitioned into 8 sets each of size 2, with the required properties. Below we present one such partition $\{A_i : i = 1, ..., 8\}$, representing the vertices of the 4-cube as binary strings:

A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
0000	1000	0100	0010	0001	0011	0110	0101
1111	0111	1010	1001	1100	1110	1101	1011

Proposition 1 $m(n) \leq \sqrt{n2^n} + 1$.

Proof.
Since the n-cube has $n2^{n-1}$ edges and each edge can only realize distance one between two sets, it immediately follows that $\binom{m}{2} \leq n2^{n-1}$. ∎

Proposition 2 $m(n) \leq \left\lfloor \frac{2^n}{m(n)} \right\rfloor n + 1$.

Proof.
If one has a partition of the n-cube into m sets, then one of the sets has at most $\lfloor 2^n/m \rfloor$ elements and hence can have distance one to at most $\lfloor 2^n/m \rfloor n$ other sets. ∎

The reason for including this improved upper bound is that it gives the correct values for $n = 1, 2, ..., 4$.

Now we give a simple lower bound, which later will be improved.

Proposition 3 $m(n) \geq \sqrt{2^n}$.

Proof.
Consider the 2^{n-1} edges of the n-cube having a fixed direction, i.e., which connect vertices having some (say the first) $n-1$ entries equal. These edges are disjoint and partition the vertices of the n-cube. Now if $\binom{m}{2} \leq 2^{n-1}$ there are enough edges to create distance one between all the m sets, because we can label the endpoints of the edges independently of one another.

To be specific, we can consider any system of disjoint sets $B_i \subseteq \{0,1\}^{n-1}$ of cardinality $|B_i| = m - i$ ($i = 1, ..., m-1$), label the vertices in $B_i \times \{0\}$ (in anyway) with $i+1, ..., m$ and in $B_i \times \{1\}$ with i. The remaining vertices in the n-cube can be labeled by any number from 1 to m. ∎

Remark 1 *Instead of the partition above one can also partition the vertices of the n-cube in squares (that is 2-cubes). Now take $m/2$ squares and label a pair of diagonal vertices for all of them with 1 and 2. The other pairs of diagonal vertices of these squares we label with $(1,2), (3,4), ..., (m-1, m)$. At this step we have that the sets given by labels 1 and 2 have distance 1 with all other sets. Further, take another $(m-2)/2$ squares and label one pair of diagonal vertices of them with 3 and 4, and label similarly to above the other diagonal vertices with $(3,4), ..., (m-1, m)$. Continuing in such a way one can get a lower bound $m(n) \geq \sqrt{2^{n+1}}$.*

For the general case we try to imitate this process: we partition the n-cube in 2^{n-k} k-cubes and try to partition the k-cube in two parts. One part will contain a fixed set of consecutive labels, starting with label l say, distributed in some way, and this will be so for a certain number of k-cubes, and the other parts will contain all labels $l, \ldots, m$ if we take these k-cubes all together. The trick is to find a way to partition the k-cube in such a way that this works and essentially produces the only edge between any pair (i, j) of subsets. As is easy to see for $k = 3$ this is not possible in general, but it can be done if k is a power of 2. This leads us to the following bound:

Theorem 1 $m(n) \geq \frac{\sqrt{2}}{2}\sqrt{n2^n}$.

2 The construction

Now we use the dual form of our problem. For a given number m of subsets we want to find the minimal dimension n of the unit cube, in which the required partition is possible.

Consider first the unit cube of dimension $k = 2^a$. Partition this k-cube in two parts, the words of even weight, and the words of odd weight. Let H be the (extended) Hamming code of length 2^k. So every word in H has even weight, H is a linear code of dimension $k - 1 - a$ and minimum distance 4.

Now partition the set of words of even weight in cosets of H. It follows that every word of odd weight has a unique neighbor in each even coset of H. Indeed, if some such word α has at least two neighbors β, γ in some coset, then the Hamming distance between β and γ is 2, but the minimum distance in the code and all its cosets is 4.

To make the description a little bit easier, we make our construction in two steps. At the first step we produce for every pair (i, j) of subsets two edges connecting them. At the second step we improve the construction by producing almost each pair (i, j) once.

Let $m = 2^{2^a-1}$ and partition the set $\{1, ..., m\}$ in 2^{2^a-a-1} groups of consecutive numbers. Therefore each group is of size 2^a, which equals the number of even cosets of H. Now for the l-th group we take a k-cube and label its points as follows. Every even coset of H gets its own fixed number from the l-th group. The words of odd weight all get different labels $1, ..., m$. Since the number of such words is exactly m, this is precisely possible. We now have realized connections between all pairs (i, j) of subsets with i in the l-th group and j arbitrary. Next we do the same for each l ($l = 1, 2, ..., 2^{2^a-a-1}$).

The total number of 2^a-cubes we use in this process equals the number of groups, i.e., 2^{2^a-a-1}. So this produces a partition of the cube of dimension $n = 2^{a+1} - a - 1$ in $m = 2^{2^a-1}$ sets and for $m/\sqrt{n2^n}$ we get

$$\frac{2^{2^a-1}}{\sqrt{(2^{a+1} - (a+1))2^{2^{a+1}-(a+1)}}},$$

and this tends to $1/2$ as $a \to \infty$.

What is not so good in this construction is that we produce exactly two edges between each pair of subsets, while if we like to reach the upper bound $\sqrt{n2^n}$, almost each pair of them should be linked by just one edge. To achieve that, choose $m = b \cdot 2^{2^a-1}$. Partition again the set $\{1, ..., m\}$ in b hypergroups of size 2^{2^a-1} of consecutive numbers. Further, partition each hypergroup in 2^{2^a-a-1} groups of size 2^a of consecutive numbers.

Consider the first hypergroup now. For the group with number l ($l = 1, ..., 2^{2^a-a-1}$) we take b subcubes of dimension 2^a and label their points as follows. Every even coset of H gets its own fixed number from the l-th group as above. The words of odd weight of the p-th subcube ($p = 1, ..., b$) get different labels from the p-th hypergroup. Now we have realized connections between pairs (i, j) of subsets with i in the first hypergroup and j arbitrary.

In the next step we consider the second hypergroup, take for each group of it $b - 1$ subcubes of dimension 2^a, and repeat the procedure in the last paragraph with the only difference that we forget about the first hypergroup now. So that way we produce edges of type (i, j) with i in the first hypergroup and j in the second only once.

In general we consider the hypergroup with number q ($q = 1, ..., b$), take for each group of it exactly $b-q+1$ subcubes of dimension 2^a and repeat with them the procedure above, labeling the odd weight vertices of them with numbers from the hypergroup with numbers $q, q + 1, ..., b$.

In such a way we get the only edge between subsets (i,j) $(i<j)$ with i,j taken from different hypergroups, and exactly two edges of type (i,j), if i,j belong to the same hypergroup. Therefore, when the number of hypergroups is large enough, we realized the only edge between almost all pairs of subsets.

The number of 2^a-cubes we used now equals

$$2^{2^a-a-1}\cdot(b+(b-1)+\cdots+1)=b(b+1)/2\cdot 2^{2^a-a-1}.$$

Let $b=2^c-1$, then $b(b-1)/2<2^{2c-1}$. Therefore, we have a partition of the unit cube of dimension $2^{a+1}-a-1+2c-1$ into $(2^c-1)\cdot 2^{2^1-a-1}$ sets, and the ratio $m/\sqrt{n2^n}$ converges to $\sqrt{2}/2$ as $c\to\infty$, $a\to\infty$, $c/a\to 0$.

3 Final remarks

The problem we considered above for the hypercube can be stated for graphs in general. So for a graph G the problem is to determine $m(G)=$ the maximal number of sets in a partitioning of the vertex set of G with the property that between two different sets there always is at least one edge. If $e=e(G)$ is the number of edges of G, then as in Proposition 1 we have

$$\binom{m}{2}\le e$$

or roughly $m<\sqrt{2e}$. Now we have shown that for the hypercube this trivial upper bound is the correct value up to a constant factor, but the question remains how special this property is. Using standard probabilistic arguments it is not hard to see that

$$\liminf\frac{m(G)}{\sqrt{2e(G)}}=0,$$

but it would still be interesting to have an explicit family of graphs G_i, $i=0,1,2,\ldots$ with $\lim\limits_{i\to\infty} m(G_i)/\sqrt{2e(G_i)}=0$. We conjecture that this is true for the Paley graphs.

References

[1] R.Ahlswede, N.Cai, Z.Zhang *Higher level extremal problems*, Preprint 92-031 of SFB 343 “Discrete Strukturen in der Mathematik”, submitted to Combinatorica.